Combinatorics of the stability space of Fine compactified Jacobians

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Motivation:

- Fine compactified Jacobians (FCJ) are defined by J. Kass, N. Pagani and O.Tommasi over nodal curves for arbitrary genus in [KP22] and [PT20].
- Classical stability conditions for FCJ have been constructed in [OS79] starting from polarisations. A polarisation ϕ is the datum of a rational number for each irreducible component of the curve such that the sum of components is 0.
- Generalised stability conditions classify FCJ in [KP22] and [PT20]. These can be described for curves of genus 1. We aim to classify general stability conditions for FCJ of curves of genus 2.

Generalised stability conditions for FCJ:

Let Γ be a connected undirected finite multigraph. Let $\{T_i\}$ be the collection of spanning trees and $k(\Gamma) = |\{T_i\}|$ be it's complexity.

A **stability condition**, σ_Γ, is the datum of an assignment,

$$\sigma_\Gamma : \{T_i\} \to \mathbb{Z}^{\mathrm{Vert}(\Gamma)}$$

such that

$$\sum_{v \in \mathrm{Vert}(\Gamma)} \sigma_\Gamma(T_i) = 0$$

and such that "if we add 1 at the endpoints of all edges in $E(\Gamma \setminus T_i)$ in all possible ways we obtain a set of $k(\Gamma)$ elements". More precisely the last condition requires the set

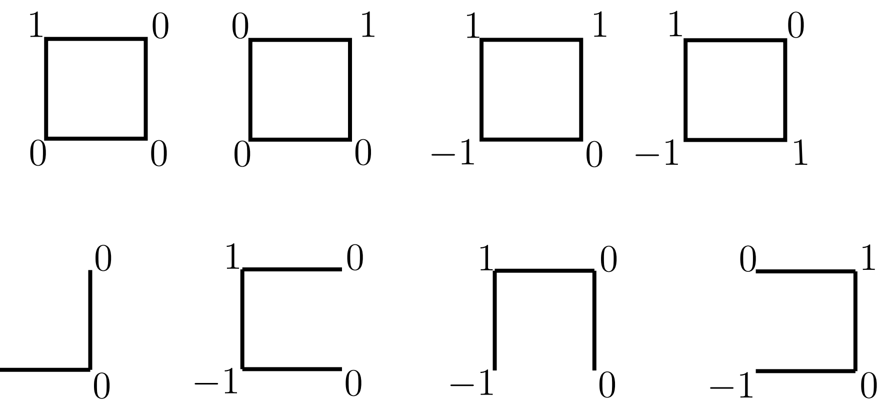
$$N_{\sigma_\Gamma} = \bigcup_{T_i} \left(\sigma_\Gamma(T_i) + \sum_{\substack{s \in \mathcal{O}(\Gamma \setminus T_i) \\ e \in \mathrm{Edges}(\Gamma \setminus T_i)}} \delta_{s(e)} \right)$$

to have $k(\Gamma)$ elements, with $\mathcal{O}(\Gamma \setminus T_i)$ the set of orientations on $\mathrm{Edges}(\Gamma \setminus T_i)$.

Remark. For ϕ generic (in an appropriate precise sense). There is a way to define a stability condition σ_Γ^ϕ, as the collection of all $d : \mathrm{Vert}(\Gamma) \to \mathbb{Z}$ that are "close enough to ϕ" (in a sense that is defined precisely in [OS79]). It can be shown that classical σ_Γ^ϕ are generalised stability conditions.

Let I_n denote a necklace graph a genus 1 made of n edges and n vertices connected in a cyclic manner.

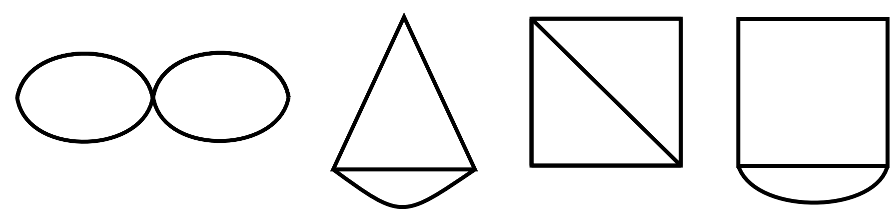
Example. We have the following stability condition for I_4,



Project:

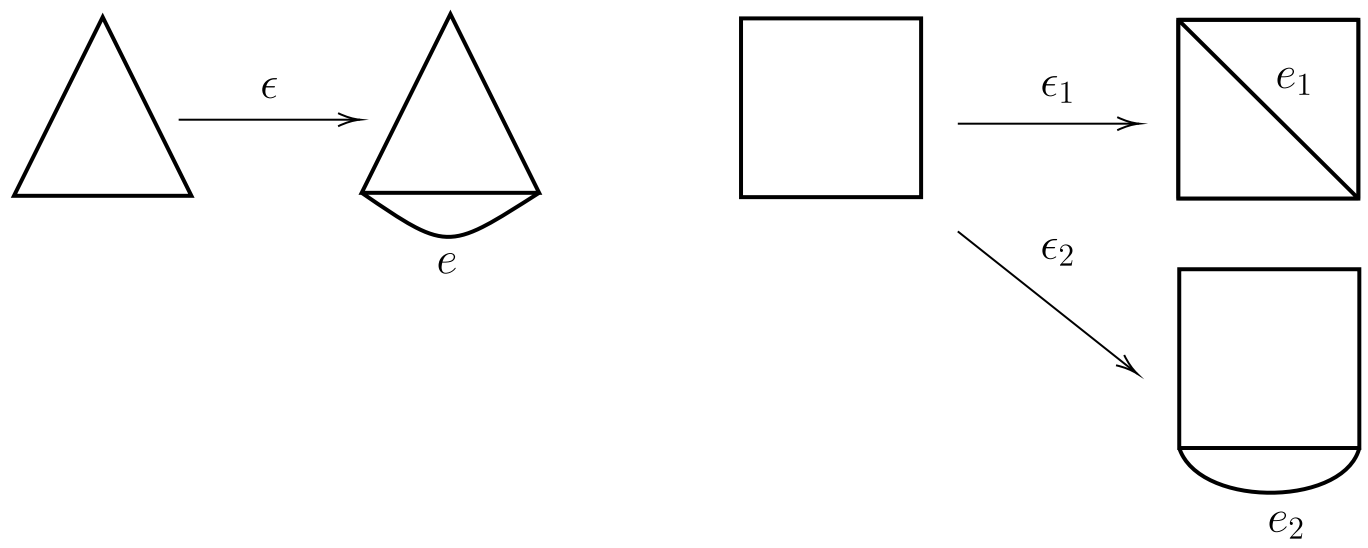
To make the collection of stability conditions finite we modulo translate and let $\vec{0} \in \sigma_\Gamma(T)$ for some $T \in \{T_i\}$. Denote the set of all σ_Γ modulo translation by Σ_Γ.

- We have classified all stability conditions Σ_Γ for Γ one of:



Each σ_Γ here is σ_Γ^ϕ by taking ϕ to be the average of all $d \in N_{\sigma_\Gamma}$.

- We have also shown we can extend uniquely from Σ_{I_3} to $\Sigma_{I_3 \cup \{e\}}$, Σ_{I_4} to $\Sigma_{I_4 \cup \{e_1\}}$, and Σ_{I_4} to $\Sigma_{I_4 \cup \{e_2\}}$.



In general we propose stability conditions, σ_Γ, for genus 2 graphs Γ are unique extensions of σ_{I_n} where $\Gamma = I_n \cup \{e\}$ for $e \notin I_n$.

Challenge:

Determine σ_Γ for each genus 2 graph Γ.

Proposed solution:

Reduce to the classification of genus 1 graphs. More precisely show we extend uniquely from Σ_{I_n} to $\Sigma_{I_n \cup \{e\}}$ for $e \notin I_n$.

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References

[KP22] Jesse Kass and Nicola Pagani. Classifying fine universal jacobian stabilities. 2022. In preparation.

[OS79] Tadao Oda and Conjeerveram S Seshadri. Compactifications of the generalized jacobian variety. *Transactions of the American Mathematical Society*, pages 1–90, 1979.

[PT20] Nicola Pagani and Orsola Tommasi. Geometry of genus one fine compactified universal jacobians. *arXiv preprint arXiv:2012.09142*, 2020.