Combinatorics of the stability space of fine compactified Jacobians

Rhys Wells

The University of Liverpool

Student Seminar 2023

Table of Contents

1 Introduction

2 Background geometry and stability conditions

3 Algorithm, analysis and conjectures

Optimal torch configurations

Suppose we have a collection of torches which light up a wall at a given distance. We call a collection of torches which light up the minimal area, a optimal torch configuration (OTC).

Optimal torch configurations

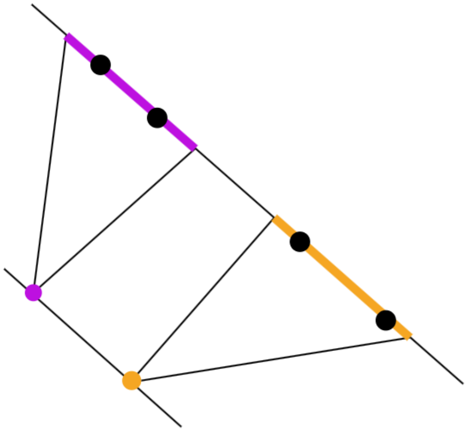
Suppose we have a collection of torches which light up a wall at a given distance. We call a collection of torches which light up the minimal area, a optimal torch configuration (OTC).

Informal questions

- If we fix a single torch, can we determine every possible OTC?
- Is there a simply way to state OTC's?

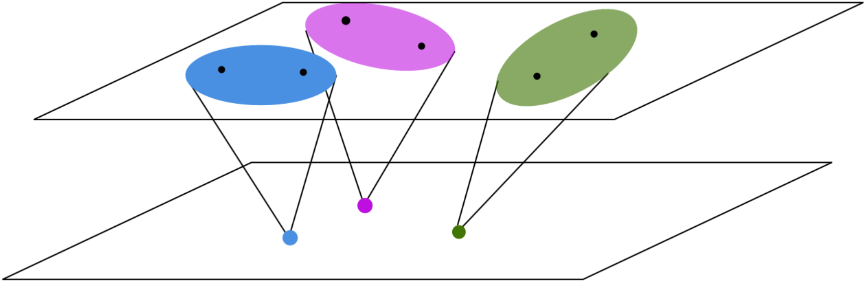
Dimension 2 example

Consider the case where we have two identical torches.



Dimension 3 example

Consider the case where we have three identical torches.



Simple remarks

- It is clear that if every torch takes the same position then the area is minimal. There are often more than this.

Simple remarks

- It is clear that if every torch takes the same position then the area is minimal. There are often more than this.
- For a configuration to be a OTC, torches in the collection must be close together. We will come back to this.

What are OTC's?

- They are data you can associated to a graph.

What are OTC's?

- They are data you can associated to a graph.
- This data classifies some object related to a curve.

What are OTC's?

- They are data you can associated to a graph.
- This data classifies some object related to a curve.
- How do you get from a curve X to its graph $\Gamma := \Gamma_X$?

What is the object which this data classifies?

- Historically one can associate to a curve X its Jacobian (which we take to be the moduli space of line bundles of degree $g(\Gamma)$).

What is the object which this data classifies?

- Historically one can associate to a curve X its Jacobian (which we take to be the moduli space of line bundles of degree $g(\Gamma)$).
- For a singular curve the Jacobian often fails to be compact. How do we compactify it?

What is the object which this data classifies?

- Historically one can associate to a curve X its Jacobian (which we take to be the moduli space of line bundles of degree $g(\Gamma)$).
- For a singular curve the Jacobian often fails to be compact. How do we compactify it?
- One way to compactify is to consider "degenerate line bundles" over X. In particular appropriate subschemes of $\mathrm{Simp}^{g(\Gamma)}(X)$.

What is the object which this data classifies?

- Historically one can associate to a curve X its Jacobian (which we take to be the moduli space of line bundles of degree $g(\Gamma)$).
- For a singular curve the Jacobian often fails to be compact. How do we compactify it?
- One way to compactify is to consider "degenerate line bundles" over X. In particular appropriate subschemes of $\mathrm{Simp}^{g(\Gamma)}(X)$.
- A fine compactified Jacobian, FCJ, of X is a connected, open and proper (i.e. compact) subscheme of $\mathrm{Simp}^{g(\Gamma)}(X)$ (that is smoothable).

What is the data?

- FCJ are constructed by taking subspace of $\mathrm{Simp}^{g(\Gamma)}(X)$ consisting of degenerate line bundles of particular multidegrees.

[1] Jesse Leo Kass and Nicola Pagani. "Classifying fine universal Jacobian stabilities". In: *Preparation* (2022).

What is the data?

- FCJ are constructed by taking subspace of $\mathrm{Simp}^{g(\Gamma)}(X)$ consisting of degenerate line bundles of particular multidegrees.
- Where the multidegree of a degenerate line bundle is a divisor $D \in \mathrm{Div}(\Gamma)$.

[1] Jesse Leo Kass and Nicola Pagani. "Classifying fine universal Jacobian stabilities". In: *Preparation* (2022).

What is the data?

- FCJ are constructed by taking subspace of $\mathrm{Simp}^{g(\Gamma)}(X)$ consisting of degenerate line bundles of particular multidegrees.
- Where the multidegree of a degenerate line bundle is a divisor $D \in \mathrm{Div}(\Gamma)$.
- It is enough to consider these divisors on the graph of the curve.

[1] Jesse Leo Kass and Nicola Pagani. "Classifying fine universal Jacobian stabilities". In: *Preparation* (2022).

What is the data?

- FCJ are constructed by taking subspace of $\text{Simp}^{g(\Gamma)}(X)$ consisting of degenerate line bundles of particular multidegrees.
- Where the multidegree of a degenerate line bundle is a divisor $D \in \text{Div}(\Gamma)$.
- It is enough to consider these divisors on the graph of the curve.
- FCJ are constructed by a packet of divisors subject to constraints, we call this a stability condition[1].

[1] Jesse Leo Kass and Nicola Pagani. "Classifying fine universal Jacobian stabilities". In: *Preparation* (2022).

Definition of a stability condition

Definition

A *stability condition** is a function

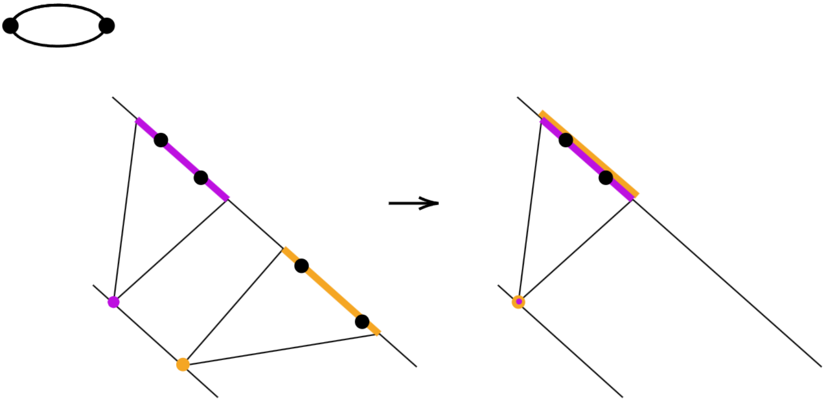
$$A_\Gamma : \mathcal{ST}(\Gamma) \to \mathrm{Div}^0(\Gamma)$$

such that

$$\sigma_\Gamma^{A_\Gamma}(\Gamma) := \bigcup_{T \in \mathcal{ST}(\Gamma)} \{A_\Gamma(T) + \sum_{e \in E(\Gamma) \setminus E(T)} \delta_{s(e)} \mid s \in \mathcal{O}(\Gamma \setminus T)\}$$

and $|\sigma_\Gamma^{A_\Gamma}(\Gamma)| = |\mathcal{ST}(\Gamma)|$.

Vine graph revisited



Summary

Analogy	Combinatorics
Distance to wall	$g(\Gamma)$
Position of torches	$A_\Gamma : \mathcal{ST}(\Gamma) \to \mathrm{Div}^0(\Gamma)$
Torch light area	$\{A_\Gamma(T) + \sum_{e \in \mathrm{E}(\Gamma \setminus T)} \delta_{s(e)} \mid s \in \mathcal{O}(\Gamma \setminus T)\}$
Total light area	$\sigma_\Gamma^{A_\Gamma}(\Gamma)$

FCJ by ϕ-stability conditions i.e Linearity

- Classical stability conditions for FCJ have been constructed starting from polarisations[2]. Where a polarisation $\phi \in \mathbb{R}^{b_1(\Gamma)}$ i.e a rational number for each vertex of the graph that the sum to $g(\Gamma)$ [3].

[2] Tadao Oda and Conjeerveram S Seshadri. "Compactifications of the generalized Jacobian variety". In: *Transactions of the American Mathematical Society* (1979), pp. 1–90.

[3] In particular ϕ has to be generic (satisfy some set of inequalities with respect to ϕ and Γ).

FCJ by ϕ-stability conditions i.e Linearity

- Classical stability conditions for FCJ have been constructed starting from polarisations[2]. Where a polarisation $\phi \in \mathbb{R}^{b_1(\Gamma)}$ i.e a rational number for each vertex of the graph that the sum to $g(\Gamma)$ [3].
- There is a way to define a ϕ-stability condition, as the collection of all divisors that are "close enough to ϕ".

[2] Tadao Oda and Conjeerveram S Seshadri. "Compactifications of the generalized Jacobian variety". In: *Transactions of the American Mathematical Society* (1979), pp. 1–90.

[3] In particular ϕ has to be generic (satisfy some set of inequalities with respect to ϕ and Γ).

FCJ by ϕ-stability conditions i.e Linearity

- Classical stability conditions for FCJ have been constructed starting from polarisations[2]. Where a polarisation $\phi \in \mathbb{R}^{b_1(\Gamma)}$ i.e a rational number for each vertex of the graph that the sum to $g(\Gamma)$ [3].
- There is a way to define a ϕ-stability condition, as the collection of all divisors that are "close enough to ϕ".
- In particular we have,

$$\{\phi\text{-stability conditions}\} \subseteq \{\text{stability conditions}\}$$

[2] Tadao Oda and Conjeerveram S Seshadri. "Compactifications of the generalized Jacobian variety". In: *Transactions of the American Mathematical Society* (1979), pp. 1–90.

[3] In particular ϕ has to be generic (satisfy some set of inequalities with respect to ϕ and Γ).

FCJ by ϕ-stability conditions i.e Linearity

- Classical stability conditions for FCJ have been constructed starting from polarisations[2]. Where a polarisation $\phi \in \mathbb{R}^{b_1(\Gamma)}$ i.e a rational number for each vertex of the graph that the sum to $g(\Gamma)$ [3].
- There is a way to define a ϕ-stability condition, as the collection of all divisors that are "close enough to ϕ".
- In particular we have,

$$\{\phi\text{-stability conditions}\} \subseteq \{\text{stability conditions}\}$$

- We have the opposite inclusion for vine graphs (seen previously) and for genus 1 graphs.

[2] Tadao Oda and Conjeerveram S Seshadri. "Compactifications of the generalized Jacobian variety". In: *Transactions of the American Mathematical Society* (1979), pp. 1–90.

[3] In particular ϕ has to be generic (satisfy some set of inequalities with respect to ϕ and Γ).

Restating our questions

Let us rephrase the questions we had for OTC in terms of stability conditions for a fixed Γ.

Restating our questions

Let us rephrase the questions we had for OTC in terms of stability conditions for a fixed Γ.

Questions

1. Can we determine every possible stability condition up to translation?
2. Is every stability condition given by a ϕ?

Restating our questions

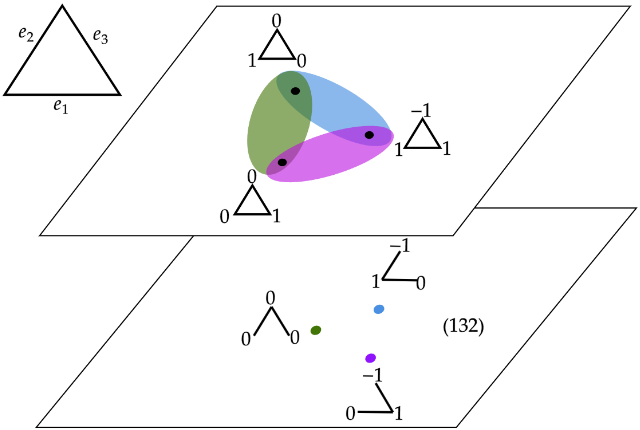
Let us rephrase the questions we had for OTC in terms of stability conditions for a fixed Γ.

Questions

1. Can we determine every possible stability condition up to translation?
2. Is every stability condition given by a ϕ?

This is simple for trees (constant) and vine graphs (ϕ given by the average). What about genus 1 graphs?

Triangle graph revisited



Genus 1 result

Let Γ be any* genus 1 graph.

- Note the trees of a genus 1 graph are the same.

Genus 1 result

Let Γ be any* genus 1 graph.

- Note the trees of a genus 1 graph are the same.
- It can be shown using the inclusion/exclusion principle that for Γ the "Torch light areas" of any OTC/stability condition overlap in a cycle.

Genus 1 result

Let Γ be any* genus 1 graph.

- Note the trees of a genus 1 graph are the same.
- It can be shown using the inclusion/exclusion principle that for Γ the "Torch light areas" of any OTC/stability condition overlap in a cycle.
- Furthermore specifying a $T \in \mathcal{ST}(\Gamma)$, a $D_T \in \mathrm{Div}^0(\Gamma)$ and a permutation τ of the edges of Γ is enough to construct a stability condition A_Γ. That is (T, D_T, τ_Γ) defines a stability condition A_Γ.

Genus 1 result

Let Γ be any* genus 1 graph.

- Note the trees of a genus 1 graph are the same.
- It can be shown using the inclusion/exclusion principle that for Γ the "Torch light areas" of any OTC/stability condition overlap in a cycle.
- Furthermore specifying a $T \in \mathcal{ST}(\Gamma)$, a $D_T \in \text{Div}^0(\Gamma)$ and a permutation τ of the edges of Γ is enough to construct a stability condition A_Γ. That is (T, D_T, τ_Γ) defines a stability condition A_Γ.
- It can also be shown that every stability condition A_Γ is given by a ϕ defined by the average of $\sigma_\Gamma^{A_\Gamma}(\Gamma)$ (the same as in the vine graph case).

Stability conditions in higher genus graphs

- For graphs of higher genus the spanning trees are not necessarily so structured.

Stability conditions in higher genus graphs

- For graphs of higher genus the spanning trees are not necessarily so structured.
- Therefore persisting with the inclusion-exclusion principle to prove results is not useful.

Stability conditions in higher genus graphs

- For graphs of higher genus the spanning trees are not necessarily so structured.
- Therefore persisting with the inclusion-exclusion principle to prove results is not useful.
- So how do we get stability conditions for higher genus graphs?

Stability conditions in higher genus graphs

- For graphs of higher genus the spanning trees are not necessarily so structured.
- Therefore persisting with the inclusion-exclusion principle to prove results is not useful.
- So how do we get stability conditions for higher genus graphs?
- We use what we know!

Stability conditions from genus 1 graphs

Let $\Gamma_{(1)}$ denote the set of connected spanning genus 1 subgraphs of Γ. In addition to the genus 1 result we have two more facts:

Stability conditions from genus 1 graphs

Let $\Gamma_{(1)}$ denote the set of connected spanning genus 1 subgraphs of Γ. In addition to the genus 1 result we have two more facts:

1. That $\bigcup_{\Gamma_0 \in \Gamma_{(1)}} \mathcal{ST}(\Gamma_0) = \mathcal{ST}(\Gamma)$,

Stability conditions from genus 1 graphs

Let $\Gamma_{(1)}$ denote the set of connected spanning genus 1 subgraphs of Γ. In addition to the genus 1 result we have two more facts:

1. That $\bigcup_{\Gamma_0 \in \Gamma_{(1)}} \mathcal{ST}(\Gamma_0) = \mathcal{ST}(\Gamma)$,and
2. for $\Gamma_0 \in \Gamma_{(1)}$ stability conditions $A_{\Gamma_0} : \mathcal{ST}(\Gamma_0) \to \mathrm{Div}^0(\Gamma_0)$ that agree on common spanning trees glue together to a function $A_\Gamma : \mathcal{ST}(\Gamma) \to \mathrm{Div}^0(\Gamma)$.

Stability conditions from genus 1 graphs

Let $\Gamma_{(1)}$ denote the set of connected spanning genus 1 subgraphs of Γ. In addition to the genus 1 result we have two more facts:

1. That $\bigcup_{\Gamma_0 \in \Gamma_{(1)}} \mathcal{ST}(\Gamma_0) = \mathcal{ST}(\Gamma)$,and
2. for $\Gamma_0 \in \Gamma_{(1)}$ stability conditions $A_{\Gamma_0} : \mathcal{ST}(\Gamma_0) \to \mathrm{Div}^0(\Gamma_0)$ that agree on common spanning trees glue together to a function $A_\Gamma : \mathcal{ST}(\Gamma) \to \mathrm{Div}^0(\Gamma)$.

- Note the total area of light given by the function A_Γ may not be minimal, we must check this!

Stability conditions from genus 1 graphs

Let $\Gamma_{(1)}$ denote the set of connected spanning genus 1 subgraphs of Γ. In addition to the genus 1 result we have two more facts:

1. That $\bigcup_{\Gamma_0 \in \Gamma_{(1)}} \mathcal{ST}(\Gamma_0) = \mathcal{ST}(\Gamma)$,and
2. for $\Gamma_0 \in \Gamma_{(1)}$ stability conditions $A_{\Gamma_0} : \mathcal{ST}(\Gamma_0) \to \mathrm{Div}^0(\Gamma_0)$ that agree on common spanning trees glue together to a function $A_\Gamma : \mathcal{ST}(\Gamma) \to \mathrm{Div}^0(\Gamma)$.

- Note the total area of light given by the function A_Γ may not be minimal, we must check this!
- After fixing a $T \in \mathcal{ST}(\Gamma)$ and $D_T \in \mathrm{Div}^0(\Gamma)$, we can use these facts to describe a method to exhaustively construct all such functions, and therefore all stability conditions up to translation.

Algorithm to construct a single stability condition

The following runs until a function

$$A_\Gamma : \mathcal{ST}(\Gamma) \to \mathrm{Div}^0(\Gamma)$$

is well-defined.

Algorithm to construct a single stability condition

The following runs until a function

$$A_\Gamma : \mathcal{ST}(\Gamma) \to \mathrm{Div}^0(\Gamma)$$

is well-defined.

1. Take $T \in \mathcal{ST}(\Gamma)$ and $D_T \in \mathrm{Div}^0(\Gamma)$.

Algorithm to construct a single stability condition

The following runs until a function

$$A_\Gamma : \mathcal{ST}(\Gamma) \to \mathrm{Div}^0(\Gamma)$$

is well-defined.

1. Take $T \in \mathcal{ST}(\Gamma)$ and $D_T \in \mathrm{Div}^0(\Gamma)$.
2. For each $\Gamma_0 \subseteq \Gamma_{(1)}$ which contains T, choose a cycle τ_{Γ_0} and generate A_{Γ_0} using $(T, D_T, \tau_{\Gamma_0})$.

Algorithm to construct a single stability condition

The following runs until a function

$$A_\Gamma : \mathcal{ST}(\Gamma) \to \mathrm{Div}^0(\Gamma)$$

is well-defined.

1. Take $T \in \mathcal{ST}(\Gamma)$ and $D_T \in \mathrm{Div}^0(\Gamma)$.
2. For each $\Gamma_0 \subseteq \Gamma_{(1)}$ which contains T, choose a cycle τ_{Γ_0} and generate A_{Γ_0} using $(T, D_T, \tau_{\Gamma_0})$.
3. Ensure the functions A_{Γ_0} agree on common spanning trees.

Algorithm to construct a single stability condition

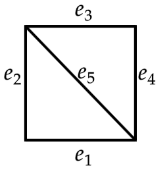
The following runs until a function

$$A_\Gamma : \mathcal{ST}(\Gamma) \to \mathrm{Div}^0(\Gamma)$$

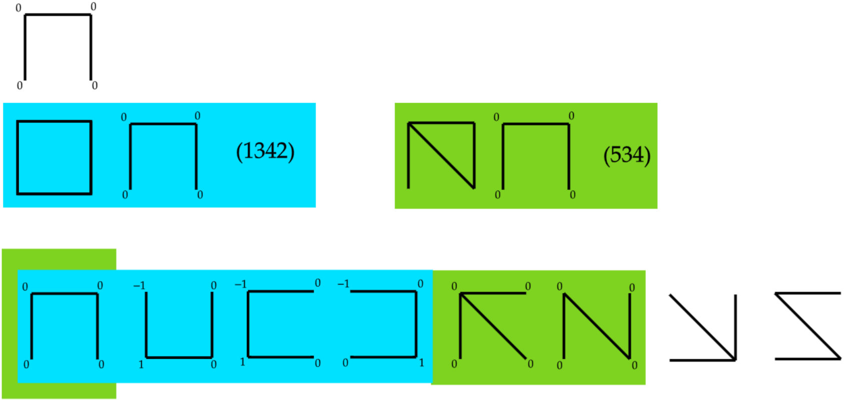
is well-defined.

1. Take $T \in \mathcal{ST}(\Gamma)$ and $D_T \in \mathrm{Div}^0(\Gamma)$.
2. For each $\Gamma_0 \subseteq \Gamma_{(1)}$ which contains T, choose a cycle τ_{Γ_0} and generate A_{Γ_0} using $(T, D_T, \tau_{\Gamma_0})$.
3. Ensure the functions A_{Γ_0} agree on common spanning trees.
4. Repeat the previous steps for a different T' and $D_{T'} = A_{\Gamma_0}(T')$ which you know.

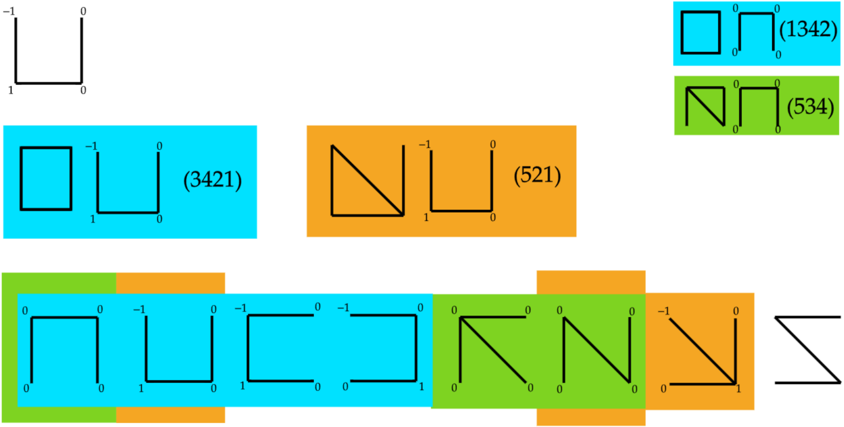
Algorithm example



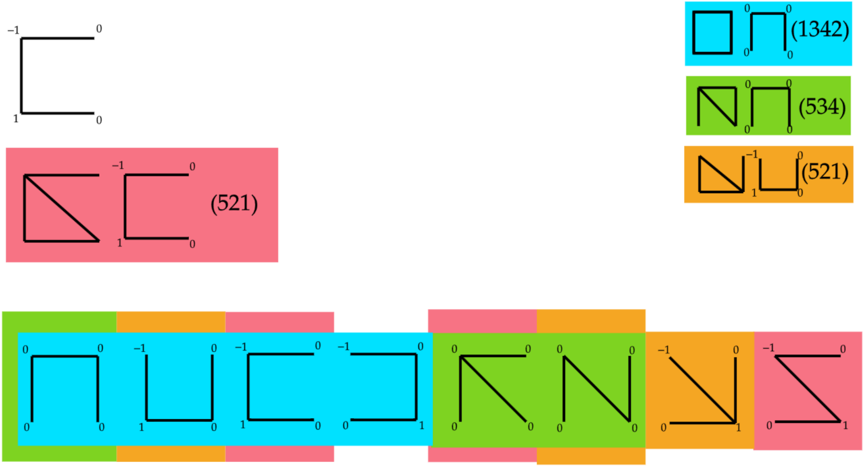
T_1



T_2

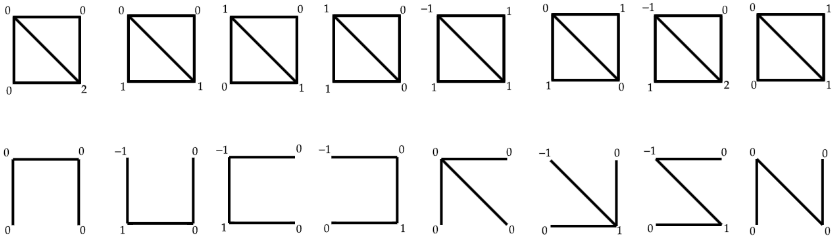


T_3



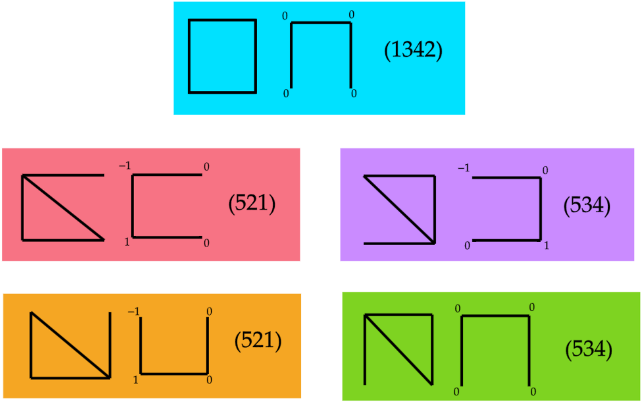
A stability condition

Now that we have A_Γ, we must check that $|\sigma_\Gamma^{A_\Gamma}(\Gamma)| = |\mathcal{ST}(\Gamma)|$.



A stability condition from genus 1

The stability condition is given by the following data.



Algorithm to find all stability conditions up to translation

- In the previous example I told you what the cycles where. How did I know this?

[4] Rhys Wells. *Stability conditions fine compactified Jacobians*. https://github.com/rhyswells101/Stability_conditions_fine_compactified_jacobians. [Online; accessed 14-June-2023]. 2023.

Algorithm to find all stability conditions up to translation

- In the previous example I told you what the cycles where. How did I know this? Iteration!

[4] Rhys Wells. *Stability conditions fine compactified Jacobians*. https://github.com/rhyswells101/Stability_conditions_fine_compactified_jacobians. [Online; accessed 14-June-2023]. 2023.

Algorithm to find all stability conditions up to translation

- In the previous example I told you what the cycles where. How did I know this? Iteration!
- 1) Choose a tree, 2) choose cycles, 3) ensure compatibility, 4) repeat 1-3) till done.

[4] Rhys Wells. *Stability conditions fine compactified Jacobians*. https://github.com/rhyswells101/Stability_conditions_fine_compactified_jacobians. [Online; accessed 14-June-2023]. 2023.

Algorithm to find all stability conditions up to translation

- In the previous example I told you what the cycles where. How did I know this? Iteration!
- 1) Choose a tree, 2) choose cycles, 3) ensure compatibility, 4) repeat 1-3) till done.
- Each time to finish 3) you have a list of possible functions to then do 1)-3) again with.

[4] Rhys Wells. *Stability conditions fine compactified Jacobians*. https://github.com/rhyswells101/Stability_conditions_fine_compactified_jacobians. [Online; accessed 14-June-2023]. 2023.

Algorithm to find all stability conditions up to translation

- In the previous example I told you what the cycles where. How did I know this? Iteration!
- 1) Choose a tree, 2) choose cycles, 3) ensure compatibility, 4) repeat 1-3) till done.
- Each time to finish 3) you have a list of possible functions to then do 1)-3) again with. This is time consuming but is exhaustive[4].

[4] Rhys Wells. *Stability conditions fine compactified Jacobians.* https://github.com/rhyswells101/Stability_conditions_fine_compactified_jacobians. [Online; accessed 14-June-2023]. 2023.

Question 2)

Now that we have a method to construct all stability conditions up to translation for a given graph Γ. We can ask the following.

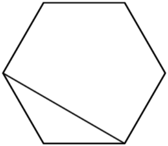
Question 2)

Now that we have a method to construct all stability conditions up to translation for a given graph Γ. We can ask the following.

- Is every stability condition A_Γ, given by ϕ the average of $\sigma_\Gamma^{A_\Gamma}(\Gamma)$ as in the vine and genus 1 graph cases?

No. Finding an explicit ϕ is hard

Consider the following graph Γ.



For the stability condition given by $A_\Gamma(T) = \vec{0}$ for all $T \in \mathcal{ST}(\Gamma)$, taking ϕ to be the average of $\sigma_\Gamma^{A_\Gamma}(\Gamma)$ fails to describe A_Γ.

Work around

- In general for a given stability condition it is hard to explicitly describe the ϕ that gives the stability condition.

Work around

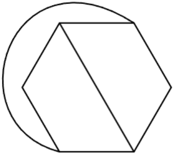
- In general for a given stability condition it is hard to explicitly describe the ϕ that gives the stability condition.
- To bypass this issue we simply ask, if the region, R_{A_Γ}, where the ϕ terms live, is empty or non-empty.

Work around

- In general for a given stability condition it is hard to explicitly describe the ϕ that gives the stability condition.
- To bypass this issue we simply ask, if the region, R_{A_Γ}, where the ϕ terms live, is empty or non-empty.
- This is something we can compute with Sagemath from a known stability condition.

First non-linear stability condition

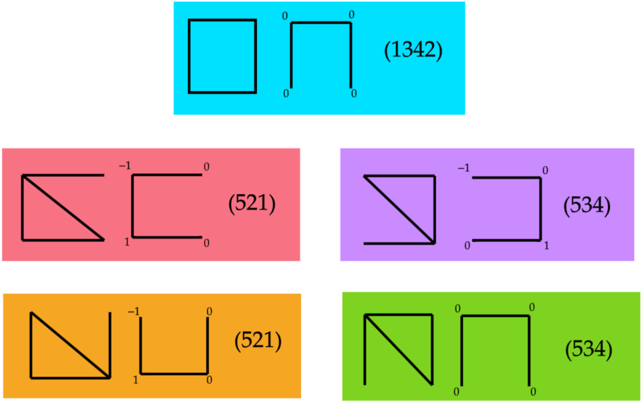
The first graph to be found with a stability condition which is not given by a ϕ is,



This occurs because the stability condition requires that ϕ must satisfy $\phi_i < \phi_j$ and $\phi_i > \phi_j$ for some i, j, a contradiction[5].

[5] Filippo Viviani. *On a new class of fine compactified Jacobians of nodal curves*. 2023. arXiv: 2310.20317 [math.AG].

Finally recall



Conjectures

- Applying this algorithm to a range of graphs one sees that any function $A_\Gamma : \mathcal{ST}(\Gamma) \to \mathrm{Div}^0(\Gamma)$ obtained by gluing together stability conditions of genus 1 subgraphs is always stability condition (we didn't need to do the check).

Conjectures

- Applying this algorithm to a range of graphs one sees that any function $A_\Gamma : \mathcal{ST}(\Gamma) \to \mathrm{Div}^0(\Gamma)$ obtained by gluing together stability conditions of genus 1 subgraphs is always stability condition (we didn't need to do the check). Is this true in general?

Conjectures

- Applying this algorithm to a range of graphs one sees that any function $A_\Gamma : \mathcal{ST}(\Gamma) \to \text{Div}^0(\Gamma)$ obtained by gluing together stability conditions of genus 1 subgraphs is always stability condition (we didn't need to do the check). Is this true in general?
- As stability condition always has compatible cycles on genus 1 subgraphs, is there a structure which can describe this set of data simply, similar to the polarisation ϕ?